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Polynomial Approximation Using Projections Whose Kernels Contain the Chebyshev Polynomials

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Let P be any projection of $f \in C^{(n+1)}[-1, 1]$ onto $\mathscr{P}_n$ such that $P(T_{n+1}) = 0$. It is shown that, for a wide class of such projections, $\|f - Pf\|$ may be expressed in terms of $f^{(n+1)}$ in the same manner as $E_n(f)$, the error of minimax approximation. Furthermore, a general necessary condition is obtained for this phenomenon.

1. INTRODUCTION

Suppose $f \in C^{(n+1)}[-1, 1]$. Then it is known that for minimax polynomial approximation on $[-1, 1]$,

$$E_n(f) := \min_{p \in \mathscr{P}_n} \|f - p\| = \frac{1}{2^n(n+1)!} |f^{(n+1)}(\xi)|, \tag{1.1}$$

where $\xi \in (-1, 1)$ and $\|\cdot\|$ denotes the Chebyshev norm on $[-1, 1]$.

It is also well known that near minimax approximation is given by the projection P onto $\mathscr{P}_n$ interpolating f at the zeros of T_{n+1}, the Chebyshev polynomial of degree $n+1$. We then have

$$\|f - Pf\| = \frac{1}{2^n(n+1)!} |f^{(n+1)}(\zeta)|, \tag{1.2}$$

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where $\zeta \in (-1, 1)$. Details of both these results may be found, for example, in [4]. Furthermore, it has recently been shown [1, 3, 5] that (1.2) also holds (for possibly different ζ, of course) if P is taken to be the projection onto $\mathscr{P}_n$ formed by truncating the Chebyshev series for f and if P is chosen such that $f - Pf$ equioscillates on the point set consisting of the $n+2$ extrema of T_{n+1}.

In this paper we first note that in all these cases of near minimax approximation $P(T_{n+1}) = 0$ and observe that this is a necessary condition for (1.2) to hold for an arbitrary projection P onto $\mathscr{P}_n$. We will prove that both of the following conditions (1.3) are necessary for (1.2) to hold.

(i) $PT_{n+1} = 0$; (1.3(i))

(ii) For each $t \in [-1, 1]$, Pf_t must interpolate $f_t(x) = (x-t)_+^n$ at (counting multiplicities) $n+1$ points $x_i \in [z_i, z_{i+1}]$, $i = 0, 1, \ldots, n$, where $\{z_i\}_{i=0}^{n+1}$ are the consecutive extrema of T_{n+1}. (1.3(ii))

Note 1.1. In the case $n = 0$ interpret (1.3(ii)) to mean that $0 \leqslant Pf_t \leqslant 1$ and in the case $n = 1$ interpret an interpolation point of multiplicity two (at $z_1 = 0$) to mean $(f_t - Pf_t)(z_1) = 0$ and $f_t - Pf_t \geqslant 0$.

Furthermore, although we are concerned in this paper with the action of P on $f \in C^{(n+1)}[-1, 1]$, we assume that the natural domain of P is $C[-1, 1]$. Also P is automatically defined on functions which are only piecewise continuous and right continuous at the "jump points" on $[-1, 1]$, e.g., $\chi_{[t, 1]}$ $(= f_t$ if $n = 0)$.

Notation. Let $\mathscr{A} = \{P; P$ satisfies $(1.3)\}$ and $\mathscr{A}^* = \{P \in \mathscr{A}; P$ satisfies $(1.2)\}$.

We will show in Section 2 that (1.3) is necessary for (1.2) to hold and in Section 4 that $\mathscr{A}^*$ constitutes at least a large subclass of $\mathscr{A}$.

2. Proof of Necessary Conditions and Preliminary Results

THEOREM 2.1. *Condition* (1.3(i)) *is necessary for* (1.2) *to hold for all* $f \in C^{(n+1)}[-1, 1]$.

Proof. Let L be the unique polynomial in $\mathscr{P}_{n+1}$ such that $PL = 0$ and $L^{(n+1)} \equiv 1$ (i.e., $(n+1)!L$ is monic). Then $\|L - PL\| = \|L\| \geqslant 2^{-n}/(n+1)!$ with equality if and only if $L = T_{n+1}/(n+1)!$ by the well-known minimal property of T_{n+1}. ∎

Note 2.1. If $Pf \in \mathscr{P}_n$ for $f \in C^{(n+1)}[-1, 1]$, then

$$\|f - Pf\| \geqslant \frac{1}{2^n(n+1)!} \min_{-1 \leqslant x \leqslant 1} |f^{(n+1)}(x)|. \tag{2.1}$$

This follows from (1.1) and $\|f - Pf\| \geqslant E_n(f)$.

Note 2.2. Consider

$$|f(x) - (Pf)(x)| \leqslant U(x)\, |f^{(n+1)}(\zeta_x)|, \tag{2.2}$$

where $U(x) = \sup_{R \in \mathscr{R}} |R(x)|$, $\mathscr{R} = \{R : R \in C^{(n+1)}[-1, 1],\ PR = 0, \|R^{(n+1)}\| \leqslant 1\}$, is the smallest allowable value in (2.2). U is the upper envelope of $\mathscr{R}$ (see also [2]). It is clear from (2.1) that (1.2) holds if and only if

$$U(x) \leqslant \frac{1}{2^n(n+1)!} \qquad \text{for all } x \in [-1, 1]. \tag{2.3}$$

From the Taylor series with integral remainder we have

$$(f - Pf)(x) = \int_{-1}^{1} K(x, t)\, f^{(n+1)}(t)\, dt, \tag{2.4}$$

where K is the Peano kernel

$$K(x, t) = (f_t(x) - (Pf_t)(x))/n!$$

$$f_t(x) = (x - t)_+^n = \begin{cases} (x-t)^n, & x \geqslant t \\ 0, & x < t. \end{cases}$$

Thus

$$U(x) = \int_{-1}^{1} |K(x, t)|\, dt, \tag{2.5}$$

and $U \in C[-1, 1]$.

THEOREM 2.2. *Condition* (1.3(ii)) *is necessary for* (1.2) *to hold for all* $f \in C^{(n+1)}[-1, 1]$.

Proof. If (1.2) holds then $PT_{n+1} = 0$ and from (2.4)

$$\int_{-1}^{1} K(x, t)\, dt = \frac{T_{n+1}(x)}{2^n(n+1)!}.$$

Also, from (2.3) and (2.5) we have

$$\int_{-1}^{1} |K(x, t)|\, dt \leqslant \frac{1}{2^n(n+1)!}.$$

Hence in the x, t-plane along a line $x = z_i$, an extremum of T_{n+1}, $K(z_i, t)$ must not change sign for $t \in [-1, 1]$ and

$$\operatorname{sign} \int_{-1}^{1} K(z_i, t)\, dt = \operatorname{sign} T_{n+1}(z_i) = (-1)^{n+1-i}$$

and thus (except for zeros)

$$\operatorname{sign} K(z_i, t) = (-1)^{n+1-i}, \qquad i = 0, 1, ..., n+1.$$

For a given t, the error

$$e(x) := f_t(x) - (Pf_t)(x) = n!\, K(x, t)$$

must alternate in sign on $z_0, z_1, ..., z_{n+1}$. A repeated zero of e is a multiple interpolation point.

In the case $n = 1$, if $e(0) = 0$ with $t = z_1 = 0$, this root is considered to be of multiplicity two. In the case $n = 0$, f_t is not continuous at $x = t$ ($t > -1$), but $e(-1)$ and $e(1)$ are of opposite signs. ▮

A projection P from $C^{(n+1)}[-1, 1]$ onto $\mathscr{P}_n$ can be identified with an $(n+1)$-dimensional subspace $[\mathscr{L}_0, ..., \mathscr{L}_n]$ of the dual of $C^{(n+1)}[-1, 1]$, where $\mathscr{L}_0, ..., \mathscr{L}_n$ is an arbitrary basis, by $Pf = \sum_{i=0}^{n} (\mathscr{L}_i f) p_i$, where $p_i \in \mathscr{P}_n$, $0 \leqslant i \leqslant n$, and the $\{p_i\}_{i=0}^{n}$ are chosen bidual to $\{\mathscr{L}_i\}_{i=0}^{n}$, i.e., $\mathscr{L}_i p_j = \delta_{ij}$ $(0 \leqslant i, j \leqslant n)$. (For a general characterization of functionals $\mathscr{L}$ in the dual of $C^{(n+1)}[-1, 1]$, see, e.g., [2].) In the following let "supp μ" denote the support of μ and "$X \leqslant Y$" mean $x \in X$, $y \in Y$ implies $x \leqslant y$.

DEFINITION 2.1. Suppose that P can be written as $P = [\mathscr{L}_0, ..., \mathscr{L}_n]$, where each $\mathscr{L}_i$ is represented by a nonnegative Borel measure μ_i (i.e., $\mathscr{L}_i f = \int_{-1}^{1} f(t)\, d\mu_i(t)$) such that if $j \neq i$ then either $\operatorname{supp} \mu_i \leqslant \operatorname{supp} \mu_j$ or $\operatorname{supp} \mu_i \geqslant \operatorname{supp} \mu_j$. We will say that P is *positive-separated* (or simply positive if $n = 0$).

THEOREM 2.3. *If P is positive-separated, then Pf interpolates f at (counting multiplicities) $n+1$ points.*

Proof. Let $[a_i, b_i]$ be the smallest interval containing $\operatorname{supp} \mu_i$ and suppose without loss that $\int_{-1}^{1} d\mu_i(t) = 1$. Then $\mathscr{L}_i f = \int_{-1}^{1} f(t)\, d\mu_i(t) = f(x_i) \int_{-1}^{1} d\mu_i(t) = f(x_i)$ for some $x_i \in [a_i, b_i]$. Thus $\mathscr{P}_n \ni Pf$ interpolates f at (counting multiplicities) the $n+1$ points $\{x_i\}_{i=0}^{n}$. ▮

Examples of Positive-Separated Projections. (i) P is any interpolating projection. Note that this is a special case of

(ii) P is any projection where $\mathscr{L}_i = (e_{x_i} + e_{x_{i+1}})/2$ (e_x denotes point evaluation at x; i.e., $e_x(f) = f(x)$), where $-1 \leqslant x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant x_{n+1} \leqslant 1$. E.g., $x_i = z_i$, the ith extremum of T_{n+1}, $0 \leqslant i \leqslant n+1$, yields the projection P of [5] such that $f - Pf$ equioscillates on $\{z_i\}_{i=0}^{n+1}$.

Notation. As an important set of examples of $P \in \mathscr{A}$, we introduce $\mathscr{S}$, the subclass of $\mathscr{A}$ consisting of positive-separated projections P such that "supp $\mathscr{L}_i$" $=$ supp $\mu_i \subset [z_i, z_{i+1}]$, where $\{z_i\}_{i=0}^{n+1}$ are the consecutive extrema of T_{n+1}. Let $\mathscr{S}^*$ denote the subclass of $\mathscr{S}$ for which (1.2) holds.

In Section 3 we determine necessary and sufficient conditions for (1.2) to hold in the case $n = 0$, where we will see that in particular $\mathscr{S}^* = \mathscr{S}$. In Section 4 we will show that for all n, $\mathscr{A}^*(\mathscr{S}^*)$ is at least a substantial subclass of $\mathscr{A}(\mathscr{S})$.

3. The Case $n = 0$ and Examples for $n = 1$

THEOREM 3.1. *In the case $n = 0$ let $P = [\mathscr{L}]$ be positive and suppose $PT_1 = 0$. Then* (1.2) *holds, i.e.,*

$$\|f - Pf\| = |f'(\zeta)|, \tag{3.1}$$

where $\zeta \in (-1, 1)$.

Proof. By Note 2.1 we need only show $\|f - Pf\| \leqslant \|f'\|$. Without loss assume $\mathscr{L}1 = \int_{-1}^{1} d\mu(x) = 1$. Then $f(x) - (Pf)(x) = f(x) - \int_{-1}^{1} f(t)\, d\mu(t) = \int_{-1}^{1} [f(x) - f(t)]\, d\mu(t) = \int_{-1}^{1} f'(\zeta_t)[x - t]\, d\mu(t)$, where ζ_t lies between x and t. Hence $|(f - Pf)(x)| \leqslant \|f'\| \int_{-1}^{1} |x - t|\, d\mu(t)$. Now use $PT_1 = \int_{-1}^{1} t\, d\mu(t) = 0$ and $\int_{-1}^{1} d\mu(t) = 1$ to obtain

$$\begin{aligned}
\int_{-1}^{1} |x - t|\, d\mu(t) &= \int_{-1}^{x} (x - t)\, d\mu(t) - \int_{x}^{1} (x - t)\, d\mu(t) \\
&= \int_{-1}^{1} (x - t)\, d\mu(t) - 2 \int_{x}^{1} (x - t)\, d\mu(t) \\
&= x - 2 \int_{x}^{1} (x - t)\, d\mu(t) = \left[1 - 2 \int_{x}^{1} d\mu(t)\right] x + 2 \int_{x}^{1} t\, d\mu(t) \\
&= \left[2 \int_{-1}^{x} d\mu(t) - 1\right] x - 2 \int_{-1}^{x} t\, d\mu(t).
\end{aligned}$$

But either $\int_x^1 d\mu(t) \leqslant \frac{1}{2}$ or $\int_{-1}^x d\mu(t) \leqslant \frac{1}{2}$; in the first case

$$\int_{-1}^1 |x-t|\, d\mu(t) \leqslant \left[1 - 2\int_x^1 d\mu(t)\right]|x| + 2\int_x^1 |t|\, d\mu(t)$$
$$\leqslant 1 - 2\int_x^1 d\mu(t) + 2\int_x^1 d\mu(t) = 1;$$

in the second case

$$\int_{-1}^1 |x-t|\, d\mu(t) \leqslant \left[1 - 2\int_{-1}^x d\mu(t)\right]|x| + 2\int_{-1}^x |t|\, d\mu(t)$$
$$\leqslant 1 - 2\int_{-1}^x d\mu(t) + 2\int_{-1}^x d\mu(t) = 1. \quad \blacksquare$$

The following example shows that (3.1) does not hold in general if $P = [\mathscr{L}]$, where $\mathscr{L}$ is signed (i.e., μ is a signed measure).

EXAMPLE 3.1. Let $t \in (0, 1]$ and $a > 0$ and consider $\mathscr{L} = -(a/2t)\, e_{-t} + (1 + a/t)\, e_0 - (a/2t)\, e_t$. Then $\mathscr{L}1 = 1$ and $PT_1 = \mathscr{L}x = 0$. Yet for

$$R(x) = \begin{cases} -x + a & \text{if } -1 \leqslant x \leqslant 0 \\ x + a & \text{if } 0 \leqslant x \leqslant 1, \end{cases}$$

$|R'(x)| = 1$, $x \neq 0$, and $\mathscr{L}_1 R = -(a/2t)(t + a) + (1 + a/t)\, a - (a/2t)(t + a) = 0$. Thus since $\|R\| = 1 + a$, we have that $\|U\| \geqslant 1 + a > 1 = \|T_1\|$ in (2.2) and (3.1) cannot hold.

In fact Theorem 3.1 can be obtained as a corollary of the following theorem.

THEOREM 3.2. *In the case* $n = 0$, P *satisfies* (1.2) *if and only if* $Pf = f(-1) + \int_{-1}^1 f'(s)\, dv(s)$, *where* v *is some nonnegative measure satisfying* $\int_{-1}^1 dv = 1$ *and* $\int_x^1 dv \leqslant (1 - x)/2$, *for all* $-1 \leqslant x \leqslant 1$.

Proof. First, from, e.g., [2], $Pf = c_0 f(-1) + \int_{-1}^1 f'(s)\, dv(s)$ for some constant c_0 and some bounded Borel measure v. Next, $P1 = 1$ implies $c_0 = 1$. Now if P satisfies (1.2), then (1.3(i)) implies $PT_1 = Px = 0$, i.e., $0 = -1 + \int_{-1}^1 dv$, and necessary condition (1.3(ii)) (see Note 1.1) implies that for all t, $P[\chi_{[t,1]}] = \lim_{\varepsilon \to 0}(1/2\varepsilon) \int_{t-\varepsilon}^{t+\varepsilon} d\nu(s) = c(t) \in [0, 1]$ and hence ν is a nonnegative measure. Finally,

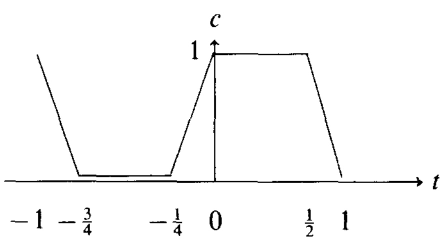
$$\begin{aligned} U(x) &= \int_{-1}^{1} |\chi_{[t,1]}(x) - P[\chi_{[t,1]}(\cdot)](x)| \, dt \\ &= \int_{-1}^{x} [1 - c(t)] \, dt + \int_{x}^{1} c(t) \, dt \\ &= \int_{-1}^{1} [1 - c(t)] \, dt - \int_{x}^{1} dt + 2 \int_{x}^{1} c(t) \, dt \\ &= x + 2 \int_{x}^{1} c(t) \, dt \\ &= x + 2 \int_{x}^{1} d\nu, \end{aligned}$$

since $\int_x^y c(t)\,dt = \int_x^y d\nu$ for all (x, y). Thus (1.2) holds if and only if $U(x) \leqslant 1$, i.e., $\int_x^1 d\nu \leqslant (1-x)/2$ for all $-1 \leqslant x \leqslant 1$. ∎

Note 3.1. Theorem 3.1 can be obtained as a corollary of Theorem 3.2, as follows: Suppose $Pf = \int_{-1}^{1} f(s)\, d\mu(s) = f(-1) + \int_{-1}^{1} f'(s)\, d\nu(s)$. Hence $f(-1) + f(s)\,c(s)|_{-1}^{1} - \int_{-1}^{1} f(s)\,c'(s)\,ds = \int_{-1}^{1} f(s)\,d\mu(s) \quad \forall f \in C^{(1)}[-1, 1]$ implies $c(1) = 0$, $c(-1) = 1$, and $-c'(s)\,ds = d\mu(s)$. Thus $U(x) = x + 2\int_x^1 c(t)\,dt$ implies $U(1) = 1 = U(-1)$ and $U''(x) = -2c'(x) \geqslant 0$. Hence $U(x)$ is concave and $U(x) \leqslant 1$, $x \in [-1, 1]$.

Note 3.2. The condition $\int_x^1 d\nu \leqslant (1-x)/2$, $-1 \leqslant x \leqslant 1$, is essential in Theorem 3.2 as seen in the following example, i.e., the necessary conditions (1.3) are not sufficient for (1.2) to hold.

EXAMPLE 3.2. Take $c(t)$ to be the piecewise linear function of the following diagram:



Take $d\nu(t) = c(t)\,dt$. Then $\int_{-1}^{1} d\nu = 1$, but $U(0) = 2\int_0^1 c(t)\,dt = \frac{3}{2}$.

EXAMPLE 3.3. Take $c(t)=\frac{1}{2}$, $-1<t<1$, $c(-1)=1$, $c(1)=0$. Then $Pf=\int_{-1}^{1} f(s)\,d\mu(s)=-\int_{-1}^{1} f(s)\,c'(s)\,ds=\frac{1}{2}[f(1)+f(-1)]$ and $U(x)\equiv 1$, $-1\leqslant x\leqslant 1$.

Note 3.3. The analogue of the condition $\int_x^1 dv\leqslant(1-x)/2$, $-1\leqslant x\leqslant 1$ for $n\geqslant 1$, which provides sufficiency (and necessity) for (1.2) is unknown and likely very involved. We can, however, obtain a more restrictive sufficient condition which does extend (Theorem 4.3 of Section 4) to $n\geqslant 1$.

THEOREM 3.3. *Let $n=0$, and consider the set $\mathscr{A}_\varepsilon$ consisting of all the projections in $\mathscr{A}$ such that for each $t\in[-1,1]$, Pf_t must interpolate $f_t(x)=(x-t)_+^0=\chi_{[t,1]}(x)$ at a point in $[-\varepsilon,\varepsilon]$. Then $\mathscr{A}_\varepsilon\subset\mathscr{A}^*$ for $\varepsilon\leqslant\frac{1}{2}$ and $\varepsilon=\frac{1}{2}$ is largest possible.*

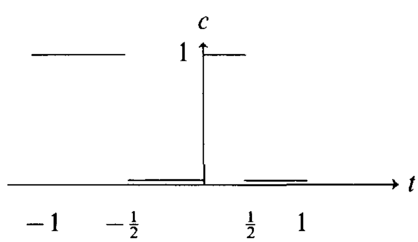
Proof.

$$P(\chi_{[t,1]})=c(t)=\begin{cases}0, & t>\varepsilon\\ 1, & t<-\varepsilon.\end{cases}$$

Then $\int_{-1}^{1} dv=\int_{-1}^{1} c(t)\,dt=1$ implies $\int_{-\varepsilon}^{\varepsilon} c(t)\,dt=\varepsilon$. Now

$$U(x)=\begin{cases}x, & x>\varepsilon\\ -x, & x<-\varepsilon\\ x+2\left[\int_x^{\varepsilon} c(t)\,dt\right], & -\varepsilon<x<\varepsilon.\end{cases}$$

Hence, for $0\leqslant x\leqslant\varepsilon$, $U(x)\leqslant x+2(\varepsilon-x)\leqslant 2\varepsilon$ since $0\leqslant c(t)\leqslant 1$ $(\forall t)$, and similarly for $-\varepsilon\leqslant x\leqslant 0$. Thus if $\varepsilon\leqslant\frac{1}{2}$, $U(x)\leqslant 1$, $-1\leqslant x\leqslant 1$. Moreover, the example



shows that $\varepsilon=\frac{1}{2}$ is largest possible. ∎

Note 3.4. Example 3.3 does not fall into $\mathscr{A}_{1/2}$ and yet it does satisfy (1.2). Thus $\mathscr{A}_{1/2}$ is not all of $\mathscr{A}^*$.

Note 3.5. Theorem 3.2 can be restated in symmetric form: $Pf=f(0)+\int_{-1}^{1} f'(s)\,dv(s)$, where $\int_{-1}^{1} dv=0$ and $\int_x^1 dv\leqslant(1-|x|)/2$, $-1\leqslant x\leqslant 1$.

THEOREM 3.4. *In the case $n=0$, the only symmetric projection ($Pf = Pf^s$, where $f^s(x)=f(-x)$) satisfying (1.3) is the projection which evaluates a function at the origin.*

Proof. It $t>0$ then by use of Notes 1.1 and 3.5 we have that $P[\chi_{[t,1]}] \in [0, 1]$ as in the proof of Theorem 3.2 and hence $v\,|_{[0,1]}$ is a positive measure. But by symmetry v is a positive measure and thus since $\int_{-1}^{1} dv = 0$, we have $v \equiv 0$. ∎

Note 3.6. The projection of Theorem 3.4 clearly satisfies (1.2). Furthermore for $n=1$, the following result can be shown.

THEOREM 3.5. *In the case $n=1$, consider a symmetric projection supported on three points $-1 \leqslant \eta_0 < \eta_1 < \eta_2 \leqslant 1$, where $\eta_1 = 0$ and $\eta_0 = -\eta_2$. Then (1.3) is sufficient for (1.2) to hold.*

The results of Theorems 3.4 and 3.5 and the results of Brass [1] lead to the following conjecture.

Conjecture. For all $n = 0, 1, 2, \ldots$ if P is a symmetric projection, then (1.3) is also sufficient for (1.2) to hold.

4. ANALYSIS OF $\mathscr{A}^*$ VIA THE PEANO KERNEL

The familiar Taylor series with integral remainder provides

$$f(x) = \sum_{i=0}^{n} f^{(i)}(-1)\,\frac{(x+1)^i}{i!} + \int_{-1}^{1} f^{(n+1)}(t)\,\frac{(x-t)_+^n}{n!}\,dt,$$

$$(x-t)_+^n = (x-t)^n,$$

if $x \geqslant t$, and $=0$, if $x<t$. Thus, for any projection P onto $\mathscr{P}_n$, $p - Pp = 0$ for $p \in \mathscr{P}_n$ and we have

$$(f - Pf)(x) = \frac{1}{n!}\int_{-1}^{1} [(x-t)_+^n - (P(\cdot - t)_+^n)]\, f^{(n+1)}(t)\,dt, \qquad (4.1)$$

the Peano kernel form of the error $(f - Pf)(x)$. Thus we have (note that, from (4.2), $U(x)$ is continuous on $[-1, 1]$)

$$U(x) = \frac{1}{n!}\int_{-1}^{1} |(x-t)_+^n - (P(\cdot - t)_+^n)|\,dt. \qquad (4.2)$$

LEMMA 4.1.

$$\frac{1}{n!}\int_{x_0}^{1} (P(\cdot - t)_+^n)(x)\,dt = (P(\cdot - x_0)_+^n)(x)/(n+1).$$

Proof. $(Pf)(x) = \sum_{i=0}^{n} [\int_{-1}^{1} f(s)\, d\mu_i(s)]\, p_i(x)$. Thus

$$\int_{x_0}^{1} (P(\cdot - t)_+^n)(x)\, dt = \int_{x_0}^{1} \sum_{i=0}^{n} \left[\int_{-1}^{1} (s-t)_+^n\, d\mu_i(s)\right] p_i(x)\, dt$$
$$= \sum_{i=0}^{n} \left[\int_{-1}^{1} \int_{x_0}^{1} (s-t)_+^n\, dt\, d\mu_i(s)\right] p_i(x)$$
$$= \sum_{i=0}^{n} \left[\int_{-1}^{1} \frac{(s-x_0)_+^{n+1}}{n+1}\, d\mu_i(s)\right] p_i(x)$$

since if $s > x_0$,

$$\int_{x_0}^{1} (s-t)_+^n\, dt = \int_{x_0}^{s} (s-t)^n\, dt$$
$$= (s-x_0)^{(n+1)}/(n+1),$$

and otherwise $\int_{x_0}^{1} (s-t)_+^n\, dt = 0$. ∎

THEOREM 4.1. *Suppose that for some fixed $x \in [-1, 1]$, the Peano kernel $k(x, t) = [(x-t)_+^n - (P(\cdot - t)_+^n)(x)]/n!$ changes sign at most once at $t_x \in [-1, 1]$ (if no change of sign, take $t_x = \pm 1$). Then*

$$U(x) = \frac{1}{(n+1)!} \,|(x+1)^{n+1} - (P(\cdot + 1)^{n+1})(x) - 2[(x-t_x)_+^{n+1} - (P(\cdot - t_x)_+^{n+1})(x)]|. \tag{4.3}$$

Proof. By Lemma 4.1,

$$U(x) = \left|\int_{-1}^{t_x} - \int_{t_x}^{1} k(x,t)\, dt\right| = \frac{1}{n!} \left|\int_{-1}^{1} - 2\int_{t_x}^{1} [(x-t)_+^n - (P(\cdot - t)_+^n)(x)]\, dt\right|$$
$$= \frac{1}{(n+1)!} \,|(x+1)^{n+1} - (P(\cdot + 1)^{n+1})(x) - 2[(x-t_x)_+^{n+1} - (P(\cdot - t_x)_+^{n+1})(x)]|. \quad ∎$$

(Theorem 4.1 coincides with [2, Theorem 9] in the case $t_x = \pm 1$.)

COROLLARY 4.1. *If L is the unique polynomial in $\mathcal{P}_{n+1}$ such that $PL = 0$ and $L^{(n+1)} \equiv 1$, and if for some fixed $x \in [-1, 1]$, $k(x, t)$ changes sign at most once at $t = t_x \in [-1, 1]$ (if no change of sign, take $t_x = \pm 1$), then*

$$U(x) = \left| L(x) - \frac{2}{(n+1)!} [(x-t)_+^{n+1} - (P(\cdot - t_x)_+^{n+1})(x)] \right|. \tag{4.4}$$

Proof. In (4.3), $(x+1)^{n+1}/(n+1)! = L + p$, where $p \in \mathscr{P}_n$. Hence

$$\frac{1}{(n+1)!}\,[(x+1)^{n+1} - (P(\cdot+1)^{n+1})(x)]$$
$$= L + p - P(L+p) = L + p - PL - Pp$$
$$= L + p - 0 - p = L. \quad \blacksquare$$

COROLLARY 4.2. *If for some fixed* $x \in [-1, 1]$, $k(x, t)$ *does not change sign* (*as a function of* t) *in* $(-1, 1)$, *then*

$$U(x) = |L(x)|. \tag{4.5}$$

Proof. Taking $t_x = -1$ or $t_x = 1$ yields $[(x - t_x)_+^{n+1} - (P(\cdot - t_x)_+^{n+1})(x)]/(n+1)! = L$ (as in the proof of Corollary 4.1) or 0, respectively. Thus (4.5) follows from (4.4). $\blacksquare$

LEMMA 4.2. *For each fixed* t, *the Peano kernel* $k(x, t) = [(x-t)_+^n - (P(\cdot - t)_+^n)(x)]/n!$ *either has* (*as a function of* x) *at most* $n+1$ *distinct roots in* $[-1, 1]$ *or else is identically zero either in* $[-1, t]$ *or in* $[t, 1]$.

Proof. For $n = 0$ see Note 1.1. So assume $n \geqslant 1$. We show by induction that for each fixed t, if $p \in \mathscr{P}_n$ then $(x-t)_+^n - p(x)$ either has at most $n+1$ distinct roots in $[-1, 1]$ or else is identically zero either in $[-1, t]$ or in $[t, 1]$. For $n = 1$ the proposition is immediate by inspection. Suppose the proposition is true for n and suppose $(x-t)_+^{n+1} - p$ is not identically zero either in $[-1, t]$ or in $[t, 1]$ and that $(x-t)_+^{n+1} - p$ has more than $n+2$ distinct roots in $[-1, 1]$. Then by Rolle's theorem $[(x+t)_+^{n+1} - p]' = (n+1)(x+t)_+^n - p'$ has more than $n+1$ distinct roots in $[-1, 1]$, contradicting the induction hypothesis, since if $(n+1)(x+t)_+^n - p'$ were identically zero in $I = [-1, t]$ or $[t, 1]$, then $(x+t)_+^{n+1} - p$ would have to be a nonzero constant in I and thus by inspection could have at most one $(< n+2)$ root in $[-1, 1]$. $\blacksquare$

Note 4.1. Note by inspection that for t fixed, if $k(x, t) \equiv 0$ in $[-1, t]$, then $k(x, t) > 0$ in $(t, 1]$, and if $k(x, t) \equiv 0$ in $[t, 1]$ then $(-1)^{n+1} k(x, t) > 0$ in $[-1, t)$.

THEOREM 4.2. *Suppose that for each* $t \in [-1, 1]$, Pf_t *must interpolate* $f_t(x) = (x-t)_+^n$ *at* (*counting multiplicities*) $n+1$ *points* x_i *in* $[a_i, b_i]$, *where* $b_{i-1} \leqslant x \leqslant a_i$, $0 \leqslant i \leqslant n+1$ $(b_{-1} = -1,\ a_{n+1} = 1)$. *Then the Peano kernel* $k(x, t)$ (*of Theorem* 4.1) *does not change sign in* $[-1, 1]$ (*as a function of* t) *for each fixed* $x \in [-1, 1] - \bigcup_{i=0}^n (a_i, b_i)$. *In fact* $(-1)^{n+1-i} k(x, t) \geqslant 0$, $0 \leqslant i \leqslant n+1$.

Proof. For each fixed t we see that $k(x, t) = f_t - Pf_t$ has at least $n+1$ zeros $x_i \in [a_i, b_i]$, $i = 0, ..., n$. Hence by Lemma 4.2 and Note 4.1, the conclusion follows. ∎

Using Corollary 4.2, we conclude the following.

COROLLARY 4.3. *Under the hypothesis of Theorem* 4.2, $U(x) = |L(x)|$ (*see Corollary* 4.1) *for* $x \in [-1, 1] - \bigcup_{i=0}^{n} (a_i, b_i)$.

Letting $a_i = b_i$, $i = 0, ..., n$, we obtain the following known result (e.g., [2, Theorem 3]).

COROLLARY 4.4. *If P is an interpolating projection, then* $U(x) = |L(x)|$ *for all* $x \in [-1, 1]$.

LEMMA 4.3. $\mathscr{A}$ *and* $\mathscr{A}^*$ *are convex.*

Proof. Let P_1, $P_2 \in \mathscr{A}$, where P_1 and P_2 satisfy (1.2). Then for all $\lambda \in [0, 1]$, $P = \lambda P_1 + (1-\lambda) P_2$ satisfies (1.3(i)) since P_1 and P_2 satisfy (1.3(i)). Also, since each function $f_t(x) = (x-t)_+^n$ is nondecreasing, it follows that since P_1 and P_2 satisfy (1.3(ii)), so does P. Hence $\mathscr{A}$ is convex. Furthermore (2.1) holds for P and if also P_1 and P_2 each satisfy (1.2), then

$$\begin{aligned} \|f - Pf\| &\leqslant \lambda \|f - P_1 f\| + (1-\lambda) \|f - P_2 f\| \\ &= \frac{1}{2^n (n+1)!} [\lambda |f^{(n+1)}(\zeta_1)| + (1-\lambda) |f^{(n+1)}(\zeta_2)|] \\ &= \frac{1}{2^n (n+1)!} |f^{(n+1)}(\zeta)| \end{aligned}$$

for some ζ between ζ_1 and ζ_2. ∎

Note (*and Notation*) 4.2. The class $\mathscr{A}$ contains for each $\varepsilon \in [0, 1]$ the set $\mathscr{A}_\varepsilon$ ($\mathscr{A} = \mathscr{A}_1$) consisting of all the projections P in $\mathscr{A}$ such that for each $t \in [-1, 1]$, Pf_t must interpolate $f_t(x) = (x-t)_+^n$ at (counting multiplicities if $\varepsilon = 1$) $n+1$ points $x_i \in [w_i - \varepsilon(w_i - z_i), w_i + \varepsilon(z_{i+1} - w_i)]$, where $\{w_i\}_{i=0}^n$ are the consecutive roots of T_{n+1}. Also denote $\mathscr{S}_\varepsilon = \mathscr{S} \cap \mathscr{A}_\varepsilon$. (Recall that by Theorem 3.1, $\mathscr{S}^* = \mathscr{S}_1$ $(= \mathscr{S})$ if $n = 0$.)

THEOREM 4.3. $\mathscr{A}^*$ (*the subclass of* $\mathscr{A}$ *for which* (1.2) *holds*) *contains* $\mathscr{A}_{\varepsilon_1}$ *for some* $\varepsilon_1 > 0$.

Proof. By Corollary 4.3, $U_\varepsilon(x) = |T_{n+1}(x)|/(n+1)!$ for all $x \in [-1, 1] - \mathscr{N}_\varepsilon$, where $\mathscr{N}_\varepsilon = \bigcup_{i=0}^{n} (w_i - \varepsilon(w_i - z_i),\ x_i + \varepsilon(z_{i+1} - w_i))$. Fix $\varepsilon < 1$. Then $U_\varepsilon(x)$ is uniformly bounded as a function of $x \in [-1, 1]$ as follows. Let g denote the mapping from $[-1,1]^2 \times \{\mathsf{X}_{i=0}^n [w_i - \varepsilon(w_i - z_i), w_i + \varepsilon(z_{i+1} - w_i)]\}$ which associates the point $(t, x, x_0, ..., x_n)$ with $(Pf_t)(x)$, the value of the polynomial which interpolates f_t on $\{x_0, x_1, ..., x_n\}$ at x. Then g is a continuous function on a compact set and hence is bounded. We conclude that $U_\varepsilon(x) = \sup(1/n!) \int_{-1}^1 |f_t(x) - (Pf_t)(x)| \, dt$ is uniformly bounded (in x).

But also $U_\varepsilon(x)$ is continuous as follows. $U_\varepsilon(x)$ is the upper envelope of a set $\mathscr{R}_\varepsilon$ consisting of $R \in C^{(n+1)}[-1,1]$ and $\|R^{(n+1)}\| \leqslant 1$. Suppose $U_\varepsilon(x)$ were discontinuous at x_0. Then there exists a sequence $x_i \to x_0$ so that $U_\varepsilon(y_i) \not\to U_\varepsilon(x_0)$. Thus, since $U_\varepsilon(x)$ is bounded on $[-1,1]$, there exists a convergent subsequence $y_i = x_{n_i}$ of x_i so that $U_\varepsilon(y_i) \to u_0 \neq U_\varepsilon(x_0)$ and suppose without loss that $u_0 < U_\varepsilon(x_0)$. Thus there must exist a sequence $R_i \in \mathscr{R}_\varepsilon$ such that $R_i(x_0) \to U_\varepsilon(x_0)$, but $R_i(y_i) \to u_1 \leqslant u_0$. Hence $R_i(x_0) - R_i(y_i) = R_i'(\zeta_i)(x_0 - y_i)$ for some sequence ζ_i between x_0 and y_i implies $|R_i'(\zeta_i)| \to \infty$. But there exists a uniform bound for $\|R'\|$, $R \in \mathscr{R}_\varepsilon$ as follows: if $n = 0$ we are done; if $n \geqslant 1$,

$$R(x) = \frac{1}{n!} \int_{-1}^{1} [(x-t)_+^n - (Pf_t)(x)] \, R^{n+1}(t) \, dt$$

implies

$$R'(x) = \frac{1}{n!} \int_{-1}^{1} [n(x-t)_+^{n-1} - (Pf_t)'(x)] \, R^{n+1}(t) \, dt$$

and we get a uniform bound for R' since the coefficients of $(Pf_t)'$ are bounded by nM_ε and thus the polynomials $(Pf_t)'(x)$ are uniformly bounded (in x and t). Thus we contradict $|R_i'(\zeta_i)| \to \infty$ and the assumption that $U_\varepsilon(x)$ is discontinuous.

Now by Corollary 4.3 for each fixed x, $U_\varepsilon(x)$ decreases (as $\varepsilon \to 0$) to $|T_{n+1}(x)|/(n+1)!$ and hence by Dini's theorem $U_\varepsilon(x)$ decreases uniformly. Thus first since $T_{n+1}(w_i) = 0$ $(i = 0, ..., n)$, pick ε_0 such that $x \in \mathscr{N}_{\varepsilon_0}$ implies $|T_{n+1}(x)| \leqslant \frac{1}{2} \|T_{n+1}\|$. Then pick $0 < \varepsilon_1 \leqslant \varepsilon_0$ such that $|U_{\varepsilon_1}(x) - |T_{n+1}(x)|/(n+1)!| \leqslant \frac{1}{2} \|T_{n+1}\|/(n+1)!$ for all $x \in [-1,1]$. We conclude that $|U_{\varepsilon_1}(x)| \leqslant \|T_{n+1}\|/(n+1)!$ for all $x \in [-1,1]$. Thus $|U(x)| \leqslant \|T_{n+1}\|/(n+1)!$ and thus (1.2) holds (recall Note 2.2) for all $P \in \mathscr{A}_{\varepsilon_1}$. ∎

Note 4.3. The largest possible allowable value of ε_1 in Theorem 4.3 is of course of interest. Only for the case $n = 0$ is the value known (see Section 3).

By Lemma 4.3 we can note finally the following fact.

COROLLARY 4.5. *If P belongs to the convex hull of $\mathscr{A}_{\varepsilon_1} \cup \{P_e\}$, where $\mathscr{A}_1 \ni P_e$ is the "equioscillating" projection of* [5] *or the truncated Chebyshev projection of* [1, 3], *then P satisfies* (1.2).

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